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Near-Geostationary Orbit Model Used in Satellite Catalog Maintenance

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An evolution model for free near-geostationary satellite orbits is presented. The procedure of averaging over the mean anomalies for the satellite, the moon, and the sun is employed to obtain the long-period evolution of the orbit element sets. Equations thus obtained are then integrated numerically with the 10–20 day time step. The gravitational influence of external bodies and zonal and tesseral harmonics of the gravitational field of the Earth, sun radiation pressure, as well as the precession and nutation of the Earth equator are taken into account while computing both long- and short-period perturbations. A list is presented of the 500 short- and long-period inequalities used in the model and discuss the optimal organization of the calculations. The presented model is now being used by the Russian Space Surveillance System.

Introduction

THE problem of studying geostationary orbits evolution has been attracting attention since the first satellite launches, and their main features are sufficiently well known by now. A variety of theories that describe the motion of geostationary satellites with different precision has been developed.

The purpose of this work is to develop a geostationary satellite motion model suitable for maintaining a catalog of near-geostationary space objects tracked by the Russian Space Surveillance System.

Geostationary satellites are tracked on the basis of series of optical measurements that are separated from each other by relatively long periods of time. This is why a sophisticated technique that establishes a correlation between these measurements and a particular satellite, and updating the orbit of the satellite with the data obtained, is necessary.[1] The need for sorting the possible distributions of these measurements over the orbits (of the already cataloged satellites) is essential in formulating the requirements for the parameters of the prediction algorithm. This need increases the number of times the algorithm is called by several orders of magnitude.

Thus, CPU time consumption becomes the main characteristic of the algorithm, whereas the accuracy requirements remain moderate because the geostationary satellites are not placed closer than $\approx$10 km to each other. This is why a prediction error significantly smaller than the preceding value will not lead to a decrease in the sorting time.

Construction of a high-speed algorithm requires solving the following two key problems: 1) choosing the factors to include and 2) choosing the averaging scheme.

Factors that determine a geostationary satellite motion include the gravitational field of the Earth, gravity of the moon and the sun, sun radiation pressure, motion of the equator with respect to the ecliptics, and so on (in our algorithm other influences are neglected as comparatively small). The following technique was used to select the necessary set of perturbation factors to include in the theory.

A special program was used to compute the coefficients of expansion for each perturbation up to a sufficiently high arbitrary order (the absolute value of each index was considered to be less then 7). The absolute value maximum for each term in the expansion was determined in the region satisfying the following conditions for the semimajor axis $\boldsymbol{a}$, eccentricity $\boldsymbol{e}$, and inclination $\boldsymbol{i}$ of the orbit: 35,000 km $\leq a \leq$ 50,000 km, $0 \leq e \leq 0.1$, and $0 \leq i \leq \pi/6$. For each orbital element, the terms of the expansions with the amplitudes below 0.2 arc-s or 10^{-7} rad/day for the periodic and secular/long-period terms, respectively, were chosen. We found $\approx$500 such terms. (Note that this means that about 1 term has been chosen among the 50 comprising the initial expansion.)

For these perturbations, high-performance programs were written and manually optimized. Same parts of different formulas were computed only once, and out of many expressions for the same value, for example, an inclination function, the fastest one was selected.

Most authors[2–4] prefer single averaging over the fast variable of a satellite as the averaging scheme. This method is simple and more than one order of magnitude faster than numerical integration. It is common to employ a polynomial representation of the moon coordinates over a 4-day time interval for such a technique. Thus, the step of the numerical integration of the averaged equations is taken to be $\approx$0.1 of the 4-day period, that is, a fraction of the satellite orbital period.

To increase the integration step for the averaged equations, we average over the mean anomalies of each of the perturbing bodies, in addition to the common averaging over the fast variable, the mean anomaly of the satellite.

This allows the use of a 10–20 day integration step and, consequently, a decrease in the required CPU time manifold (compared to using coordinates of the moon from the an annual astronomical table) while analyzing the sets of observations that are separated by long time intervals (a typical practice in our system).

Nonsingular Orbital Element Sets

The following system of the orbital element sets is used in the algorithm:

$$t_0, \lambda = M + \omega + \Omega, \qquad L = \sqrt{\mu a}$$
$$p = \sigma \cdot \cos\Omega, \qquad q = \sigma \cdot \sin\Omega$$
$$k = e \cdot \cos\pi, \qquad h = e \cdot \sin\pi$$

where

a = semimajor axis
e = eccentricity
i = inclination
M = mean anomaly
$\boldsymbol{S}$ = matching ballistic coefficient
t_0 = epoch of the elements
μ = gravitational constant
π = $\Omega + \omega$
σ = $\sin(i/2)$
Ω = longitude of ascending node
ω = perigee argument

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General Description of the Method

The main frame of reference $OXYZ$ is Cartesian and is fixed at the time t_0. If we neglect precession of the axis of the Earth for the time period involved, we can obtain any current frame of reference $Oxyz$ by rotating $OXYZ$ through an angle $\varphi = \varpi_{a^\circ} \cdot (t - t_0)$ about the axis of the Earth, where ϖ_{a°: is the angular velocity of the Earth.

The Hamiltonian of the problem is

$$H = H_0 + H_{\text{per}} \tag{1}$$

where

$$H_0 = -[(\boldsymbol{V})^2/2] + \mu/R = \mu/2L^2 \tag{2}$$

is the part of the Hamiltonian corresponding to the unperturbed motion, where

H_{per} = perturbing potential consisting of three parts
H_1 = perturbing potential of the zonal harmonics
H_2 = perturbing potential of the tesseral harmonics
H_3 = perturbing potential of the moon and the sun gravity
$\boldsymbol{R}$ = position vector of the satellite
$\boldsymbol{V}$ = velocity of the satellite

Equations of the Perturbation Theory

For the Delaunay elements

$$\boldsymbol{p} = (L_1, L_2, L_3), \qquad \boldsymbol{q} = (l_1, l_2, l_3)$$

the Zeipel–Brouwer method allows a nearly identical transformation excluding fast variables pertaining to the motion along the orbits of the satellite, the moon, and the sun from the Hamiltonian

$$(\boldsymbol{p}, \boldsymbol{q}) \Rightarrow (\boldsymbol{p}', \boldsymbol{q}')$$

The preceding substitution is performed with the generating function

$$S(\boldsymbol{p}', \boldsymbol{q}, t)$$

according to the following formulas:

$$\boldsymbol{p} = \frac{\partial S}{\partial \boldsymbol{q}}, \qquad \boldsymbol{q}' = \frac{\partial S}{\partial \boldsymbol{p}'} \tag{3}$$

The Hamiltonian and the generating function are expanded into series with a small parameter as an argument.

The first-order Zeipel–Brouwer theory yields

$$-\frac{\partial S}{\partial t} - n\frac{\partial S}{\partial l} + H(\boldsymbol{p}', \boldsymbol{q}, \psi_{\text{ex.b}}, t) = H^*(\boldsymbol{p}', \hat{\boldsymbol{q}}, \hat{\psi}_{\text{ex.b}}, t) \tag{4}$$

where

n = average angular velocity of the satellite, μ^2/L^3
$\psi_{\text{ex.b}}$ = parameters of the orbits of external bodies contained in the Hamiltonian
$*$ = averaged Hamiltonian
$\hat{}$ = vectors lacking fast variables

The composition of the vectors $\hat{\boldsymbol{q}}$, $\hat{\psi}_{\text{ex.b}}$ will be different for different perturbations.

Where $i = 1$, the zonal harmonic perturbations are independent of time. The slow variables vector will be $\hat{\boldsymbol{q}} = (-, l_2, l_3)$, and the perturbations are determined in their final form without expansion in powers of the eccentricity.

Where $i = 2$, the time dependence of the tesseral harmonics is included through the sidereal time. The Hamiltonian is then expanded in powers of the eccentricity. Relevant to the averaged Hamiltonian are the resonance terms $\hat{\boldsymbol{q}} = \{l_1 - \varpi_e \cdot t, l_2, l_3\}$.

Where $i = 3$, for the perturbations from external bodies, time dependence is incorporated through both the mean anomaly of the influencing body and through the so-called fundamental arguments of the Hill–Brown theory (this, of course, is only true if the external bodies are represented by the moon). There are certain difficulties in expanding the Hamiltonian with this dependence explicit. The case when equatorial elements of the moon are functions of the slow fundamental arguments will be considered later.

Thus, for the variables $\boldsymbol{p}'$ and $\boldsymbol{q}'$, we obtain Hamilton's equations with the averaged function H^*:

$$\dot{\boldsymbol{p}}' = \frac{\partial H^*}{\partial \boldsymbol{q}'}, \qquad \dot{\boldsymbol{q}}' = -\frac{\partial H^*}{\partial \boldsymbol{p}'}$$

These equations are integrated numerically after the computation of the appropriate partial derivatives with respect to the elements.

Perturbations Related to the Oblateness of the Earth

Because the perturbations caused by the influence of the zonal harmonics of Earth gravity are well known and extensively documented, they are not discussed in this paper. Our algorithm utilizes the finite Kozai formulas.[5]

Perturbations pertaining to the influence of the tesseral harmonics of Earth gravity depend on the satellite position longitude, which is practically constant for a truly geostationary orbit. Thus, there is a resonance between the rotation of the Earth and the satellite motion, which alters this motion significantly. The standard Hamiltonian expansion is used to study the effects of the tesseral harmonics:

$$H_{\text{tes}} = \sum_{n=2}^{5}\sum_{m=1}^{5} H_{n,m} \tag{5}$$

where

$$\Re_{n,m} = (i^{n-m})\alpha_n \sum_{r=-n}^{n}\sum_{s=-\infty}^{\infty} A_{n,m}^{(r)} X_{-n-1,r}^{(s)} (C_{n,m} - i \cdot S_{n,m})$$
$$\times \exp i[s \cdot M + r \cdot \omega + m \cdot (\Omega - S_{\text{sid}})] \tag{6}$$

$$H_{n,m} = \text{Re}\{\Re_{n,m}\} \tag{7}$$

In Eqs. (5–7), the following designations are used:

$A_{n,m}^{(r)}$ = inclination function
$C_{n,m}$, $S_{n,m}$ = constants of the gravitational field
i = $\sqrt{(-1)}$
$\Re_{n,m}$ = complex function with the real part equal to $H_{n,q}$
S_{sid} = sidereal time
$X_{-n-1,r}^{(s)}$ = eccentricity function (Hansen polynomial)
α_n = coefficient, $\mu/a \cdot (R_e/a)^n$

Table 1 Resonance inequalities

Number	Indices				Classes		
	m	n	r	s	1	2	3
1	1	3	1	1	λ	L, q	k
2	1	3	3	1	——	——	λ
3	1	3	−1	1	——	λ	——
4	1	4	0	1	——	λ	——
5	1	4	2	1	——	λ	——
6	1	5	1	1	——	λ	——
7	2	2	0	2	——	λ	L, k
8	2	2	2	2	λ, L	k, q	——
9	2	3	1	2	λ	——	L, k, q
10	2	3	3	2	λ	——	L, k, q
11	2	4	0	2	——	λ	L
12	2	4	2	2	λ	——	L, q
13	2	4	4	2	——	——	λ
14	2	5	1	2	——	λ	——
15	3	3	1	3	λ	——	——
16	3	3	3	3	λ	——	——
17	3	4	0	3	——	——	λ
18	3	4	2	3	λ	——	L, k
19	3	4	4	3	λ	——	——
20	3	5	1	3	——	——	λ
21	3	5	3	3	λ	——	λ
22	4	4	2	4	——	λ	——
23	4	4	4	4	λ	——	L
24	4	5	3	4	——	λ	——
25	4	5	5	4	——	λ	——
26	5	5	3	5	——	λ	——
27	5	5	5	5	λ	——	——

As already mentioned, the averaged function will include resonance angles:

$$\theta = m \cdot (M - S_{\text{sid}})$$

That is why the index s is constrained, and only four independent indices remain. For each type of perturbation in Tables 1–4 we present the terms included in the expansions. After the ordinal number in the first column, in the next columns we give the set of indices characterizing the respective term of the expansion; under the heading Classes we give the orbital elements, with respect to which the partial differentiating of the Hamiltonian is performed. This operation yields the perturbations of orbital elements produced by this term of the Hamiltonian. In these designations we take into account that some variables are coupled, and the symbol q means computing both q and p, whereas k corresponds to k and h. The meaning of division into the classes will be explained later. Table 1 lists in order all of the long-period resonance inequalities employed in the algorithm.

Note that the minimum power of the eccentricity is $\|s - r\|$. The long-period perturbations are obtained by integrating the corresponding partial derivatives of the perturbing Hamiltonian (in this case, H_2).

Table 2 Short-period inequalities

	Indices				Classes		
Number	s	n	m	r	1	2	3
28	1	2	0	0	—	—	λ, L, q, k
29	1	3	0	1	—	—	—
30	1	3	1	1	—	—	—
31	2	2	0	0	—	—	k
32	2	2	2	2	L	—	λ, q, k
33	2	3	2	1	—	—	L
34	2	3	2	3	—	—	L
35	2	4	2	2	—	—	L
36	2	4	3	2	—	—	L
37	3	2	0	2	—	—	k, q
38	3	3	0	1	—	—	q
39	3	3	0	3	—	—	q
40	3	3	3	3	—	L	λ, q, k
41	3	4	3	2	—	—	L

Table 3 Long-period inequalities pertaining to the gravity of the moon

	Indices			Classes		
Number	n	m	r	1	2	3
42	2	0	0	L, k	—	—
43	2	0	2	k	L, q	—
44	2	1	0	L, q	—	—
45	2	1	2	k, q	L	—
46	2	1	−2	—	—	L, k, q
47	2	2	0	L, q	k	—
48	2	2	2	k	—	q
49	3	0	1	—	k	L, q
50	3	1	1	—	L, k, q	—
51	3	1	−1	—	—	L, k, q
52	3	2	1	—	k	L
53	3	2	−1	—	—	L, k, q
54	3	2	3	—	—	k, q
55	3	2	−1	—	—	—
56	3	3	1	—	—	L, k, q
57	4	0	0	—	—	k
58	4	0	2	—	—	L, k, q
59	4	1	0	—	L, k, q	—
60	4	1	2	—	k	L, q
61	4	1	−2	—	—	L, k, q
62	4	2	0	—	q	L
63	4	2	2	—	—	L, k, q
64	4	3	0	—	—	L, q
65	4	3	2	—	—	L, k, q

Table 4 Short-period inequalities pertaining to the gravity of the moon

	Indices					Classes		
Number	s	s_1	n	m	r	1	2	3
66	0	1	2	0	0		L, q	k
67	0	1	2	1	0		L, q	k
68	0	1	2	1	2	—	—	L, k
69	0	1	2	2	0	—	—	L, k
70	0	1	2	2	2	—	—	L, k
71	0	1	3	0	1	—	—	L, k, q
72	0	1	3	1	1	—	—	L, k, q
73	0	1	3	1	−1	—	k	L, q
74	0	1	3	2	1	—	—	k, q
75	0	1	3	2	−1	—	—	L, k, q
76	0	2	2	0	0	—	L	k, q
77	0	2	2	1	0	—	L, k, q	
78	0	2	2	1	2	—	—	L, k, q
79	0	2	2	2	2	—	q	L
80	0	2	3	1	1	—	—	L, k, q
81	0	2	3	2	1	—	—	L, k
82	0	2	4	1	0	—	—	L, q
83	0	2	4	2	0	—	—	L, q
84	0	3	2	0	0	—	—	L, k, q
85	0	3	2	1	0	—	—	L, k
86	0	3	2	1	2	—	—	L, k
87	0	3	2	2	0	—	—	L, k
88	0	3	2	2	2	—	—	L, k
89	0	3	2	1	1	—	—	L, k, q
90	0	3	2	2	1	—	—	L, k, q
91	0	3	2	3	1	—	—	L, k, q
92	0	3	3	3	3	—	—	k
93	0	4	2	1	0	—	—	L, q
94	0	4	2	2	0	—	—	L, k
95	0	4	2	2	2	—	—	k, q
96	0	4	3	2	1	—	—	k
97	0	−1	2	1	0	—	L, k, q	—
98	0	−1	2	1	2	—	—	k
99	0	−1	3	0	1	—	—	k, q
100	0	−1	3	1	1	—	—	L, k, q
101	0	−1	3	1	−1	—	—	L, k, q
102	0	−2	2	1	0	—	—	L, q
103	1	0	2	0	0	—	—	L, k
104	1	0	2	0	2	—	—	k, q
105	1	0	2	1	0	—	—	L, k
106	1	0	2	1	2	—	—	L, k, q
107	1	0	2	2	2	—	—	L, k
108	1	1	2	2	2	—	—	k
109	1	1	3	1	1	—	—	L, q
110	1	1	3	2	1	—	—	L, q
111	1	2	2	1	0	—	—	L, k, q
112	1	2	2	1	2	—	—	L, q
113	1	2	2	2	2	—	L, k	λ
114	1	3	2	1	2	—	—	L, k
115	1	3	2	1	1	—	—	L
116	1	3	3	2	1	—	—	L, q
117	1	3	3	3	1	—	—	L
118	1	4	2	2	2	—	—	L, k
119	2	0	2	0	2	—	—	L, q
120	2	0	2	1	2	—	—	L, q
121	2	0	2	2	2	—	—	L
122	2	1	2	2	2	—	—	L
123	2	2	2	1	2	—	—	λ, L, q
124	2	2	2	2	2	—	—	L
125	2	3	2	1	2	—	—	L, q
126	2	3	2	2	2	—	—	λ, L, q
127	2	3	3	3	3	—	—	L, k
128	2	4	2	2	2	—	—	L
129	3	2	2	2	2	—	—	λ, L, q
130	3	3	3	3	3	—	—	λ, L
131	3	4	3	3	3	—	—	L
132	−1	0	2	1	0	—	—	L, k
133	−1	2	2	1	0	—	—	L, k

Listed in Table 2 are the short-period inequalities included in accordance with our criteria. Here an additional column **s** is introduced for the mean anomaly multiplicity.

The short-period perturbations are obtained by differentiation of the generating function using Eq. 3.

Perturbations due to Attraction of External Bodies

We are going to consider only the perturbations originating from the gravity of the moon because this is the most difficult case. Perturbations caused by the sun are obtained from the earlier case by substituting the appropriate values of the constants and skipping certain inequalities.

Equation (8) shows the well-known expansion of the Hamiltonian employed for this task:

$$H^{\Diamond} = (\mu^{\Diamond}/a^{\Diamond})(a/a^{\Diamond})^n \Sigma\Sigma\Sigma\Sigma\{H_{nmrs}\} \quad (8)$$

where

$A_{nm}^{(r)}$ = inclination function for the satellite

X_{nr}^{s} = eccentricity function for the satellite

$Z_{nm}^{(\Diamond)}$ = complex spherical function of the external body

$\Diamond$ = external body

$$Z_{nm}^{\Diamond} = P_n^{(m)}(\varphi_{\Diamond}) \times \exp i(m \cdot \lambda_{\Diamond}) \quad (9)$$

where $\varphi_{\Diamond}$ and $\lambda_{\Diamond}$ are the equatorial latitude and longitude of the external body

$$H_{nmrs} = A_{nm}^{(r)} \cdot X_{nr}^{s} \cdot Z_{nm}^{(\Diamond)} \times \exp + i\{s \cdot \lambda + (r-s) \cdot \pi + (m-r) \cdot \Omega\}$$

To obtain the explicit dependence of the $Z_{n,m}^{\Diamond}$ on the mean anomaly of the perturbing body, we use the following expansion:

$$Z_{n,m}^{\Diamond} = \sum_{r_1} \sum_{s_1} A_{-n-1,m}^{(r_1)} X_{n,r_1}^{(s_1)} \times \exp\{-i[s_1 \cdot \lambda_{\Diamond} + (r_1 - s_1) \cdot \pi_{\Diamond} + (m - r_1) \cdot \Omega_{\Diamond}]\} \quad (10)$$

The general expression after the summation sign in Eq. (8) can be divided into two multipliers, each of which is a function of the coordinates of the satellite and the external body. Table 3 displays, in an ordered fashion, all of the long-period inequalities related to the moon that are included in the algorithm. Appropriate multipliers that express the spherical functions of the moon are computed taking into account the slow evolution of the orbital parameters of the moon. A simplified model is employed to depict the motion of the moon as described subsequently.

Simplified Model for Motion of the Moon

The average longitude of the ascending node for the moon orbit in the ecliptic is computed as $\bar{\Omega}_{\text{moon}} = \bar{\Omega}_0 + \dot{\bar{\Omega}} \cdot (t - t_0)$, where the angular velocity is a linear combination of the velocities of the fundamental arguments from the theory of the moon motion: The equatorial element set of the moon is calculated as

$$\bar{\Omega} = \bar{\Omega}_0 + \dot{\bar{\Omega}} \cdot (t - t_0)$$

$$\Omega = \bar{\Omega} + k_{\Omega}^1 \cdot \sin(\bar{\Omega}_{\text{moon}}) + k_{\Omega}^2 \cdot \sin(2\bar{\Omega}_{\text{moon}}) + k_{\Omega}^3 \cdot \sin(3\bar{\Omega}_{\text{moon}}) + k_{\Omega}^4 \cdot \sin(4\bar{\Omega}_{\text{moon}})$$

$$\bar{\pi} = \bar{\pi}_0 + \dot{\bar{\pi}}_0 \cdot (t - t_0)$$

$$\pi = \bar{\pi} + \pi_{\Omega}^1 \cdot \sin(\bar{\Omega}_{\text{moon}}) + \pi_{\Omega}^2 \cdot \sin(2\bar{\Omega}_{\text{moon}}) + \pi_{\Omega}^3 \cdot \sin(3\bar{\Omega}_{\text{moon}})$$

$$\bar{\lambda} = \bar{\lambda}_0 + \dot{\bar{\lambda}}_0 \cdot (t - t_0)$$

$$\lambda = \bar{\lambda} + \pi_{\Omega}^1 \cdot \sin(\bar{\Omega}_{\text{moon}}) + \pi_{\Omega}^2 \cdot \sin(2\bar{\Omega}_{\text{moon}}) + \pi_{\Omega}^3 \cdot \sin(3\bar{\Omega}_{\text{moon}})$$

$$i = i_0 + i_{\Omega}^1 \cdot \cos(\bar{\Omega}_{\text{moon}}) + i_{\Omega}^2 \cdot \cos(2\bar{\Omega}_{\text{moon}}) + i_{\Omega}^3 \cdot \cos(3\bar{\Omega}_{\text{moon}})$$

where a, i_0, and e are constants and $k_{\Omega}^{1,2,3,4}$, $\pi_{\Omega}^{1,2,3}$, and $i_{\Omega}^{1,2,3}$ are also constants.

Equation (10) and the moon orbit element set are used to derive the required (corresponding to preset values of n and q) expansions of the spherical functions $Z_{n,m}^{\Diamond}$.

Table 4 is an ordered presentation of all of the lunar short-period inequalities included in the theory.

Perturbations Produced by Other Factors

In addition, the algorithm takes into account perturbations originating from the solar radiation pressure, as well as from the precession and nutation of the Earth axis. The solar radiation pressure perturbation is determined assuming a constant satellite albedo and ballistic coefficient, but influence of the shadow is included. A detailed description of the method is given in the Appendix.

Perturbations from the precession and nutation are computed according to known formulas[6] and are not described here.

Organization of the Computations

Special approaches to organization of the calculations were employed to increase the computational speed.

Elbrus class computers require a long time to find an array element (several CPU cycles). This is why a decision was made to use no arrays and to store each element as a separate variable. Because we are generally dealing with six-dimensional arrays, our code acquired an exotic look with about 10 pages of variables declarations; however, the computational cost dropped by more than a factor of 10.

For a similar reason, we use no complex-variable procedures and determine all of the necessary exponential forms by explicitly finding sine and cosine of multiple angles, which are, in turn, reduced to sums of products of the main ones.

We have also undertaken considerable effort to express the needed special functions in an efficient manner and to determine the optimal order of the indices in calculating inequalities, which was to take into account common terms in different inequalities.

An efficient scheme of computing amplitudes of the inequalities taking into account rates of their changes has been developed.

1) The function $Z_{n,m}^{\Diamond}$ undergoes the slowest rate of change because it is a function of the slow fundamental arguments. For this reason, computing these amplitudes is singled out into a separate subroutine. This program calculates coefficients of the polynomial representation for each interval for a total period of several years (the length of this period of time is a parameter) for all of the necessary amplitudes. The polynomial degree varies from 1 to 4, whereas the interval represented by one polynomial is up to 100 days without losing accuracy in the prediction. The main program receives the set of coefficients required at a particular time and then uses the polynomials for further calculations.

2) The amplitudes of all fast angles are separated into three classes according to the frequency of their renewal. For each class, a threshold for simultaneous recomputing of all of the amplitudes is defined in terms of the orbit element set changes and time intervals. Constant values of these amplitudes (different for each interval) are used until the threshold is reached. The values of the thresholds are adjusted so that, for example, the third-class amplitudes are recomputed by an order of magnitude less frequently than the second-class ones. Examining Tables 1–4 reveals that the first class contains only a few terms.

3) The amplitudes of the perturbations with the same set of indices, but related to different element sets of the orbits are assigned to different classes. This results in a decrease in the computational cost without losing accuracy.

4) Three areas of initial conditions are considered, resonance, nonresonance, and border. A set of amplitudes and perturbations to be included, as well as the time steps for numerical integration of the quasi-secular equations, are assigned separately for each of the areas. The fourth-order Runge–Kutta method is employed. A typical set of integration step values is resonance area 10 days, nonresonance area 20 days, and border area (T/N), where T is the resonance period (usually several months) and N is the number of steps in one period (usually several tens).

Table 5 Results of testing algorithm accuracy[a]

Δt	Δr	Δn	Δb	Δt	Δr	Δn	Δb	Δt	Δr	Δn	Δb
100	0.7	3.9	0.6	600	1.2	1.5	5.4	1100	0.5	3.2	7.9
200	0.8	5.7	1.5	700	0.3	2.5	2.5	1200	1.5	0.7	1.8
300	1.7	5.8	1.3	800	0.6	1.9	5.4	1300	0.3	1.0	10.0
400	0.5	5.1	3.8	900	0.4	3.7	5.9	1400	1.6	5.9	2.3
500	0.3	2.6	0.2	1000	2.4	0.6	4.1	1500	0.6	3.1	10.0

[a]Here $i = 0.3$ deg, $e = 0.003$, $T = 1436$ min, $S = 1\ e{-9}$ and Δr, Δn, and Δb are in kilometers.

Results of Experimental Tests

The described method has been tested to estimate its level of accuracy and speed. The evaluation of the accuracy was conducted by determining differences between positions of a satellite obtained with the program being tested and with a standard one at various moments of time. Numerical integration of the differential equations describing satellite motion using the Bulirsh and Stoer method[7] was adopted as the standard method for the comparison. The same perturbations were introduced for both programs. Projections of the satellite position deviations onto the radius vector, binormal line, and the line perpendicular to both (which was close to the direction of the satellite velocity) were considered separately. These deviation projections are Δr, Δb, and Δn, respectively (kilometers).

Table 5 presents an example of the test results. Characteristics of the orbit involved are listed in the first row.

Other tests have also shown that the systematic prediction errors of the presented program do not exceed 38 km or 3 angular minutes for periods of time up to 1500 days (about 4 years).

The CPU time required to perform the calculations on an Elbrus-2 computer (3.5×10^6 average operations/s) obeys the following formula:

$$\Delta T = 0.003 \cdot N + 0.001 \cdot m$$

where ΔT is CPU time in seconds, N is the number of integration steps, and m is the number of measurements in a set.

Conclusions

We have presented the evolution model for a near-geostationary satellite orbit that has been adopted by the Russian Space Surveillance Center. The characteristic feature of the method is averaging over the mean anomalies of the perturbing bodies, the moon and the sun.

A special organization of calculations used in the program resulted in a significant decrease in the CPU time required in a typical surveillance regime where predictions of the orbit are required at many times (multitude of measurements conducted at different times).

The programs have been written and used on Elbrus-2 and personal computers (in C++).

The required CPU time for an IBM Pentium-100 compatible personal computer with 8 MB of RAM has been found to be

$$\Delta T = 0.0012 \cdot N + 0.00015 \cdot m$$

where ΔT is CPU time in seconds, N is the number of integration steps, and m is the number of measurements in a set.

Appendix: Solar Radiation Perturbation

Only the largest term in the Legendre expansion for the perturbing function was considered in dealing with the light pressure:

$$H^{\text{sp}} = -\varepsilon[r/(r^s/a^s)^2]\cos(\gamma)$$

where

a^s = semimajor axis of the Earth orbit
r = length of the satellite radius vector
r^s = current sun–Earth distance
γ = angle between the directions to the satellite and the sun
ε = coefficient of light pressure for a satellite, which depends on the satellite design and semimajor axis of the Earth orbit

Only quasi-secular perturbations are considered. Short-period components are neglected.

Averaging is done over the satellite motion along the orbit (integration of the perturbing function over the mean anomaly) only. No time integration (mean anomaly of the sun) is conducted.

Effects pertaining to the shadow of the Earth are taken into the account by computing extra additions to the partial derivatives of the averaged perturbing function for the light pressure (right-hand sides of the quasi-secular equations are constructed as a linear combination of the total perturbing function partial derivatives with respect to the satellite orbit element set).

Formulas for calculating the partial derivatives of the averaged perturbing function of the solar radiation pressure forces follow.

Without Effects of the Earth's Shadow

Formulas for calculating partial derivatives introduced by the integrals

$$\overline{H_\chi^{\text{sp}}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial U^{\text{sp}}}{\partial \chi}\, \mathrm{d}M$$

where $\chi = p, q, h, k$, are as follows:

$$\bar{H}_\xi^{\text{sp}} = -\varepsilon \frac{S_{g\xi}^{e^s} - 2S_n^{e^s} S_{g\xi}^{e^n}}{(r^s/a^s)^2}$$

where $\xi = h, k$,

$$\bar{H}_\zeta^{\text{sp}} = -\varepsilon \frac{S_{g\zeta}^{e^s} S_g^n + S_n^{e^s} S_g^{e^\zeta}}{(r^s/a^s)^2}$$

where $\zeta = p, q$,

$$\bar{H}^{\text{sp}} = -\varepsilon \frac{S_g^{e^s} - 2S_n^{e^s} S_g^n}{(r^s/a^s)^2}$$

where

$$S_{gh}^{e^s} = -\tfrac{3}{2}ay_e^s, \qquad S_{gh}^n = +\tfrac{3}{2}ap, \qquad S_{gk}^{e^s} = -\tfrac{3}{2}ax_e^s$$

$$S_{gk}^n = -\tfrac{3}{2}aq, \qquad S_g^{e^s} = -\tfrac{3}{2}a\left(x_e^s k + y_e^s h\right)$$

$$S_g^n = -\tfrac{3}{2}a(qk - ph), \qquad S_n^{e^s} = x_e^s q - y_e^s p + z_e^s c$$

$$S_{np}^{e^s} = -\left[y_e^s + z_e^s(p/c)\right], \qquad S_{nq}^{e^s} = x_e^s - z_e^s(q/c)$$

$$S_g^{n^p} = +\tfrac{3}{2}ah$$

and $\boldsymbol{e}^s = (x_e^s, y_e^s, z_e^s)$ is the unit vector in the direction to the sun, a is the semimajor axis of the satellite orbit, $c = \cos(i)$, i is the inclination of the satellite orbit toward the Earth equator, and p, q, h, and k is the Lagrangian element set of the satellite orbit.

The right-hand sides of the quasi-secular equations are expressed linearly through the integrals $\overline{H_\chi^{\text{sp}}}$.

Effects of the Earth's Shadow

The Earth's shadow is assumed to have a cylindrical shape. Approximate formulas are used to find points of entering and exiting from the shadow. These formulas permit high accuracy for low-eccentricity orbits.

To obtain the point of entering to and exiting from the shadow of the Earth the following designations are used: E_{1_1} is the eccentric longitude at the point of entrance and E_{1_2} is the eccentric longitude at the point of exit, where $E_1 = E + \omega + \Omega$ is the eccentric longitude of the satellite orbit, ω is the perigee argument for the satellite orbit, and Ω is the longitude of ascending node for the satellite orbit.

The sequence of trigonometric function calculations for eccentric longitudes of the satellite orbit corresponding to the points where it enters to and exits from the shadow of the Earth are $\cos(E_{1_1})$,

$\sin(E_{1_1})$, $\cos(E_{1_2})$, $\sin(E_{1_2})$, where E_{1_1} and E_{1_2} correspond to the entrance and exit points, respectively:

$$R_{0_a} = R_0/a, \qquad S = \sqrt{1 - R_{0_a}^2}, \qquad S_1 = 1/S$$

$$\eta = \sqrt{1 - e^2}, \qquad \eta_1 = 1/(1+\eta)$$

where R_0 is the radius of the Earth,

$$k_1 = k \cdot \eta_1, \qquad h_1 = h \cdot \eta_1, \qquad A = 1 - h \cdot h_1$$

$$B = 1 - k \cdot k_1, \qquad C = h \cdot k_1$$

$$C_\varphi = 2 \cdot (A \cdot q - C \cdot p)$$

$$CC_\varphi = (A - q \cdot C_\varphi) \cdot x^s + (C + p \cdot C_\varphi) \cdot y^s - c \cdot C_\varphi \cdot z^s - k \cdot S_1$$

$$S_\varphi = 2 \cdot (C \cdot q - B \cdot p)$$

$$SS_\varphi = (C - q \cdot S_\varphi) \cdot x^s + (B + p \cdot S_\varphi) \cdot y^s - c \cdot S_\varphi \cdot z^s - h \cdot S_1$$

$$D_\psi = 2 \cdot (k \cdot q - h \cdot p)$$

$$\cos(\psi) = (k - q \cdot D_\psi) \cdot x^s + (h + p \cdot D_\psi) \cdot y^s - c \cdot D_\psi \cdot z^s - S_1$$

$$\rho_1 = 1 \Big/ \sqrt{CC_\varphi^2 + SS_\varphi^2}$$

$$\cos(\varphi) = CC_\varphi \cdot \rho_1, \qquad \sin(\varphi) = SS_\varphi \cdot \rho_1$$

The condition for a space object to be shadowed by the Earth is

$$-1 \le \cos(\psi) \le 1$$

If the preceding inequality is satisfied, we calculate

$$\sin(\psi) = \sqrt{1 - \cos^2(\psi)}, \qquad \Delta E_1 = 2 \cdot [\pi - \arccos(\cos\psi)]$$

$$\cos(E_{1_1}) = \cos(\varphi) \cdot \cos(\psi) - \sin(\varphi) \cdot \sin(\psi)$$

$$\cos(E_{1_1}) = \cos(\varphi) \cdot \cos(\psi) - \sin(\varphi) \cdot \sin(\psi)$$

$$\sin(E_{1_1}) = \sin(\varphi) \cdot \cos(\psi) + \cos(\varphi) \cdot \sin(\psi)$$

$$\cos(E_{1_2}) = \cos(\varphi) \cdot \cos(\psi) - \sin(\varphi) \cdot \sin(\psi)$$

$$\sin(E_{1_2}) = \sin(\varphi) \cdot \cos(\psi) + \cos(\varphi) \cdot \sin(\psi)$$

Then additions to the partial derivatives of the averaged perturbing function are computed.

To compute additions to the partial derivatives of the averaged perturbing function, the partial derivatives sought can be introduced with the following integrals:

$$\overline{H_\chi^{\mathrm{add}}} = \int_{M_1}^{M_2} \frac{\partial H^{\mathrm{sp}}}{\partial \chi} \, \mathrm{d}M$$

where $\chi = p, q, h, k$; M_1 and M_2 are values of the mean anomaly corresponding to the points of the satellite entrance and exit from the Earth shadow,

$$\overline{H_\xi^{\mathrm{add}}} = -\varepsilon \frac{S_{\rho\xi}^{e^s} - 2S_n^{e^s} S_{\rho\xi}^n}{(r^s/a^s)^2}$$

where $\xi = h, k$,

$$\overline{H_\zeta^{\mathrm{add}}} = 2\varepsilon \frac{S_{n_\zeta}^{e^s} S_\rho^n + S_n^{e^s} S_\rho^{n^\zeta}}{(r^s/a^s)^2}$$

where $\zeta = p, q$,

$$\overline{H^{\mathrm{add}}} = -\varepsilon \frac{S_\rho^{e^s} - 2S_n^{e^s} S_\rho^n}{(r^s/a^s)^2}$$

where r^s is the current Earth–sun distance and a^s is the major semi-axis of the Earth orbit (for the motion around the sun). Values of S_y^x are computed as follows:

$$S_{\rho_\xi}^{e^s} = x_e^s x_\xi^\rho + y_e^s y_\xi^\rho, \qquad S_{\rho_\xi}^n = q x_\xi^\rho - p y_\xi^\rho$$

where $\xi = h, k$,

$$S_\rho^{e^s} = x_e^s x^\rho + y_e^s y^\rho, \qquad S_\rho^n = q x^\rho - p y^\rho$$

$$S_n^{e^s} = x_e^s q - y_e^s p + z_e^s c$$

$$S_{n_p}^{e^s} = -\left[y_e^s + z_e^s(p/c)\right], \qquad S_{n_q}^{e^s} = x_e^s - z_e^s(q/c)$$

$$S_\rho^{n^p} = -y^\rho, \qquad S_\rho^{n^q} = x^\rho$$

To obtain x_β^α and y_β^α, the following formulas are employed:

$$x^\rho = a(\bar{C}_1 - k\Delta E_1 + h\bar{A}_1)$$

$$y^\rho = a(\bar{S}_1 - h\Delta \bar{E}_1 - k\bar{A}_1)$$

$$x_k^\rho = a[-\Delta E_1 + h_1(\bar{S}_1 + k_2\bar{A}_1) - (\overline{SS} - h\overline{SB}_1)]$$

$$y_k^\rho = a[-\bar{A}_1 - k_1(\bar{S}_1 + k_2\bar{A}_1) + (\overline{CS} - k\overline{SB}_1)]$$

$$x_h^\rho = a[\bar{A}_1 + h_1(-\bar{C}_1 + h_2\bar{A}_1) + (\overline{CS} - h\overline{CB}_1)]$$

$$y_h^\rho = a[-\Delta E_1 - k_1(-\bar{C}_1 + h_2\bar{A}_1) - (\overline{CC} - k\overline{CB}_1)]$$

where

$$\Delta E_1 = E_{1_2} - E_{1_1}$$

$$k_1 = \frac{k}{1+\eta}, \qquad h_1 = \frac{h}{1+\eta}, \qquad k_2 = \frac{k}{\eta}, \qquad h_2 = \frac{h}{\eta}$$

$$\bar{C} = \int_{E_{1_1}}^{E_{1_2}} \cos(E_1)\,\mathrm{d}E_1 = \sin(E_{1_2}) - \sin(E_{1_1})$$

$$\bar{S} = \int_{E_{1_1}}^{E_{1_2}} \sin(E_1)\,\mathrm{d}E_1 = -\left[\cos(E_{1_2}) - \cos(E_{1_1})\right]$$

$$C\bar{C} = \int_{E_{1_1}}^{E_{1_2}} \cos^2(E_1)\,\mathrm{d}E_1 = \frac{1}{2}\left[\Delta E_1 + \cos(E_{1_2})\sin(E_{1_2}) - \cos(E_{1_1})\sin(E_{1_1})\right]$$

$$S\bar{S} \int_{E_{1_2}}^{E_{1_2}} \sin^2(E_1)\,\mathrm{d}E_1 = \frac{1}{2}\left\{\Delta E_1 - \left[\cos(E_{1_2})\sin(E_{1_2}) - \cos(E_{1_1})\sin(E_{1_1})\right]\right\}$$

$$\overline{CS} = \int_{E_{1_1}}^{E_{1_2}} \cos(E_1)\sin(E_1)\,\mathrm{d}E_1 = \frac{1}{2}[\sin^2(E_{1_2}) - \sin^2(E_{1_1})]$$

$$\bar{C}_1 = \int_{E_{1_1}}^{E_{1_2}} \frac{r}{a}\cos(E_1)\,\mathrm{d}E_1 = \bar{C} - (k\overline{CC} + h\overline{CS})$$

$$\bar{S}_1 = \int_{E_{1_1}}^{E_{1_2}} \frac{r}{a}\sin(E_1)\,\mathrm{d}E_1 = \bar{S} - (k\overline{CS} + h\overline{SS})$$

$$\bar{A}_1 = \frac{1}{1+\eta}\int_{E_{1_1}}^{E_{1_2}} \frac{r}{a} e\sin(E)\,\mathrm{d}E_1 = \frac{1}{1+\eta}[k\bar{S} - h\bar{C} - (k^2 - h^2)\overline{CS} - kh(\overline{CC} - \overline{SS})] = \frac{1}{1+\eta}\left\{k\bar{S} - h\bar{C} - (k^2 - h^2)\overline{CS} - kh\left[\cos(E_{1_2})\sin(E_{1_2}) - \cos(E_{1_1})\sin(E_{1_1})\right]\right\}$$

$$\overline{CB}_1 = \frac{1}{1+\eta}\int_{E_{1_1}}^{E_{1_2}} \cos(E_1)[1 - e\cos(E)]\,\mathrm{d}E_1$$

$$= \frac{1}{1+\eta}(k\overline{CC} + h\overline{CS})$$

$$\overline{SB}_1 = \frac{1}{1+\eta}\int_{E_{1_1}}^{E_{1_2}} \sin(E_1)[1 - e\cos(E)]\,\mathrm{d}E_1$$

$$= \frac{1}{1+\eta}(k\overline{CS} + h\overline{SS})$$

The final form of the perturbing function partial derivatives for the light pressure with the Earth shadow taken into account is computed with the following formulas:

$$\overline{H_\chi^{\mathrm{FINsp}}} = \overline{H_\chi^{\mathrm{sp}}} - \overline{H_\chi^{\mathrm{add}}}$$

where $\chi = p, q, h, k$.

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